Adventures in Group Theory: Rubik's Cube, Merlin's Machine, and Other Mathematical Toys*

David Joyner, Jaap Scherphuis

5-24-2007

Contents

*New puzzles

Below we collect parts of [Sch2] which will be included in the 2nd edition of the AGT book.

1 Whip-it/Puzzle Tower

This sliding piece puzzle consists of several discs stacked up into a tower, and they can rotate about a central axle. There are usually 3 such discs (called the Whip-it puzzle, or Puzzle 6) but 6 discs is also quite common in cheaper knock-offs. Along the side of the tower are six columns of sliding pieces, in six colors. The pieces can slide up or down from one disc to another because one of the white pieces is missing leaving a gap. By rotating the discs, the pieces and the gap are moved around to the other columns.

There is a version called Varikon, which has 5 colors with 5 balls of each, placed in 5 columns and 5 discs. There is also a gap but this is in the sixth disc, so one column is in effect a bit longer than the others. The same puzzle also exists in versions with 4 columns of 4 balls, and this one has been used as a promotional item for Smarties.

This puzzle is the simplest in a family of related puzzles which includes the Babylon Tower, the Nintendo Barrel, and Missing Link.

1	2	3	4	5	6
1	2	3	4	5	6
1	2	3	4	5	6
1	2	3	4	5	6
1	2	3	4	5	6
1	2	3	4	5	*

Here the colors are indicates by the numbers 1-6 and the blank square is indicated by a *.

1.1 The number of positions

On the 6 layer version there are 36 pieces if we include the space as a piece, which gives a maximum of 36! positions. This limit is not reached because:

- the pieces of each color are indistinguishable (6!55!)
- the colors are equivalent (6!)

This last factor comes about because it does not matter which color goes in which column. This leaves $36!/(6!^6 5!) = 22,251,481,138,642,910,394,240 \approx 2.2 \cdot 10^{22}$ positions.

The 3 layer version has $18!/(3!^5 2!6!) = 571,771,200$ positions.

The 5 column Varikon has $26!/5!^6 = 135,061,494,343,776$ positions, and the 4 column version has $17!/4!^5 = 44,669,625$ positions.

1.2 Solution

The tower can be solved layers, disc by disc from the bottom up, but it is just as easy to solve column by column. The latter is much easier to describe, so that is the method used here.

1. Decide which colors the columns will have. To speed things up, you could rotate some discs to partly solve some columns already.
2. Look at the column containing the gap, and determine which color the column has.
3. Find a piece of the required color in a different column. If there are none (i.e. the gap is in the completed white column, and the puzzle is not yet solved) then find any piece on the puzzle that is incorrect.
4. Slide pieces in the column up or down until the gap is in a layer next to the layer that contains the piece.
5. Rotate the layer with the gap until it lies above or below the piece.
6. Slide the piece up or down into the gap.
7. Rotate the layer back to its original position, which brings the piece into the column as required.
8. Repeat steps 2-7 until the puzzle is solved.

Note that in practice it is quicker to move not just the one layer, but all the layers above/below it as well.

2 Rashkey

This puzzle consists of two overlapping circular disks. The distance between the centers is exactly one radius length. The disks are made up of many pieces with curved sides, which allow them to be rotated any number of quarter turns. Each disk has 4 squares, 4 diamonds, and 12 triangles. As some pieces are shared between the two disks, there are all together 6 squares, 7 diamonds, and 20 triangles.

The puzzle comes in three possible color schemes:

1. The central 4 triangles of one disk are red, those of the other disk are blue, and all the rest is yellow.
2. One disk is completely yellow, and the remaining parts of the second disk are red.
3. One disk is red and the other is blue, except that the pieces which are shared between them which are yellow.

These are in order of difficulty - 1 is easiest and 3 is hardest.

Rashkey was invented by Oleg Raschkov, and patented on 2 September 1999, DE 29,904,348 U.

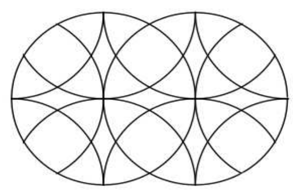

Figure 1: Rashkey puzzle

2.1 The number of positions

There are two ways of counting moves - a turn of a disc by any amount is a single move (*face turn metric*), or each quarter turn of a disc is counted as a move (*quarter turn metric*).

The triangles lie in two sets of 10 which cannot intermingle. The 33 pieces fall into sets (orbits) of 10, 10, 7, and 6 pieces which can each be mixed. There are some parity restrictions, but since all color schemes have many identical pieces, these restrictions have no discernible effect. There is a restriction on the set of squares (similar to Turn'Push), but only the coloring used in Rashkey 3 is detailed enough for it to be noticeable.

Puzzle	Positions
Rashkey 1	$(10!/4!6!)^2 = 44,100$
Rashkey 2	$(10!/2!8!)(10!/4!6!)(7!/3!4!)(6!/2!4! = 4,961,250$
Rashkey 3	$(10!/2!2!6!)^2(7!/1!3!3!) \cdot 60 = 13,335,840,000$

The number of positions for each number of moves from the start of all positions of Rashkey types 1 and 2 have been computed. The results are shown on the WWW site [Sch2] and tell us that Rashkey 1 needs at most 12 face turns (or 15 quarter turns) and Rashkey 2 at most 19 face turns (23 quarter turns). Rashkey 3 has too many positions to calculate fully.

2.2 Solution methods

Notation used in solutions: Let a clockwise quarter turn of the left disk be denoted by L. Rotations of 180, or 270 degrees are then denoted by L^2, and L^3. Turns of the right disk are denoted in the same way as R, R^2 or R^3.

2.2.1 Rashkey 1

We will use the 'the left center' to mean the four central triangles of the left disk, and similarly 'right center' are those of the right disk. We will assume that the red pieces belong in the left center when it is solved, and the blue pieces in the right center. Thus in the mixed puzzle the red pieces will be in the left center and the rim of the right disk, whereas the blue lie in the right center and the rim of the left disk. You may have to turn the puzzle upside down to achieve this.

It is actually remarkably tricky to find a short straightforward solution to this puzzle. Much of the solution below is intuitive once you see what you are trying to do, but looks complicated when written out in such detail.

Phase 1: Place all the red pieces in two pairs at the top and right of the right disk.

1. If there is not yet a pair of red pieces next to each other, then make such a pair. Now bring that pair to the top of the right disk. This is easy so we will not elaborate on it.

2. There are several possibilities for the other two red pieces. If they are together at the right hand side of the right disk, then continue with phase 2.

3. If they are together at the bottom of the right disk, then do $RL^2RL^2R^2$ to put them on the right, and then continue with phase 2.

4. If there is no red piece at the left center, then do RL^2R^3 to bring (at least) one such piece there.

5. Bring the other loose red piece to the left center as well, by turning the right disk.

6. If the two red pieces in the left center are not adjacent, then do the following steps to make put them next to each other:

   (a) Turn the left disk so that the two red pieces are at the bottom left and top right of the left center.
   (b) Turn the right disk so that there are no red pieces that lie in both disks.
   (c) Do L.
   (d) Turn the right disk to bring the two loose pieces together.

7. Turn the left disk so that its adjacent red pair lies on the left.

8. Turn the right disk to bring its pair to the bottom.

9. Do L2 R2, and the red pieces will be at the top and right of the rim of the right disk.

Phase 2: Make the two blue piece pairs on the rim of the left disk. The red piece are out of the way, so we can move the blue pieces without disturbing the reds.

1. If there is a blue piece at the top right of the right center, then you need to remove it as follows:

   (a) Make sure there is a yellow piece immediately to the left of the blue piece, by turning the left disk.
   (b) Do $R^3L^2R^2L^2R^2$.

   This swaps the order of the red pairs, dislodging the blue piece in the process.

2. Turn the left disk through 360 degrees, and as you do so check out how many of the loose blue pieces visit the top left position of the right center (suppose there are x of these) and also how many visit the bottom left position of the right center (suppose there are y of these).

3. If x is at least 2, then you can make a blue pair as follows:

   (a) Turn the left disk to bring one of the loose blue pieces to the top left of the right center.

   (b) Do R.

   (c) Turn the left disk to bring another loose blue to the top left of the right center.

   (d) Do R^3. Now go back to step b.

   (e) If y is at least 2, then you can bring together two of those pieces to make a blue pair as follows:

      i. Turn the left disk so that the two loose blue pieces do not lie at the right center, and nor does any previously solved blue pair.

      ii. Do R.

      iii. Turn the left disk to bring one of the loose blue pieces to the bottom left of the right center.

      iv. Do R^3.

      v. Turn the left disk to bring another loose blue to the bottom left of the right center.

      vi. Do R.

      vii. Turn the left disk so that the right center is completely yellow.

      viii. Do R3 to bring the red pairs to the top and right as before.

      Now go back to step b.

   (f) If x and y are both 1, then the two loose pieces cannot be paired up immediately. One of them has to be shifted as follows:

      i. Turn the left disk to bring one of the loose blue pieces to the bottom left of the right center.

      ii. Do R.

      iii. Turn the left disk so that the right center is completely yellow.

iv. Do R3.

Now go back to step c to pair them up.

Phase 3: Bring the pairs together.

1. Turn the left ring to bring one blue pair to the right center, and the other at the left or top of the left disk. The red pairs will still be at the right and top of the right disk.

2. If the second blue pair is on the left side of the left disk, then do the move sequence: R^2L^2R.

3. If the second blue pair is at the top of the left disk, then do the move sequence: $RL^3R^3LRL^3RL^2R$.

Pretty Patterns for Rashkey 1

1. Wheels 1: $R^2LR^3L^3R^2L^2RLRL^2R^3L^3$.

2. Wheels 2: $R^2LRL^3R^2L^2R^3LR^3L^2RL^3$.

2.2.2 Rashkey 2

We will assume that the red pieces belong in the left disk when it is solved. Thus in the mixed puzzle there should be more than two red triangles on the rim of the left disk. If this is not so, you have to turn the puzzle upside down.

Phase 1: Solve the 6 red triangles on the outside of the left disk. Note that you can equally well think of this phase as solving the 4 yellow triangles in the center of the right disk.

1. Turn the left disk until there are (at least) two adjacent yellow triangles in the center of the right disk, and then turn the right disk so that the two yellow triangles lie on the right of the center.

2. Turn the left disk so that there are at least three yellow pieces at the right center. If the right center is completely yellow, continue with phase 2.

3. Turn the right disk so that its single red center triangle is at the top right.

4. Turn the left disk so that the right center has two red triangles.

5. If the two red triangles in the right center are not adjacent, then

   (a) Turn the left disk so that there is only one red triangle in the right center.

   (b) Turn the right disk a quarter turn clockwise (so the center red triangle is at the bottom right)

   (c) Turn the left disk back, so the right center has two adjacent red triangles.

6. Turn the right disk so that its red center triangles are on the left.

7. Turn the left disk so that there are no red pieces in the right center.

Phase 2: Solve the 2 red triangles in the left half of the center of the left disk.

1. If the two remaining red triangles are already correct, i.e. on the left half of the left center, then go directly to the next phase.

2. Turn the right disk to bring one of the unsolved red triangles to the left center.

3. If that loose red is at the top right of the left center, then do $LR^2L^3R^2L$, but if it lies at the bottom right of the left center then do $L^3R^2LR^2L^3$. These sequences will move the loose red into the left half of the left center.

4. Turn the right disk to bring the last unsolved red triangle to the left center.

5. There are 4 cases for the last two reds, depending on where they lie in the left center.
   Bottom left and top right: $LR^2L^3R^2L$.
   Bottom left and bottom right: $R^2L^3R^2L^3R^2L^2R^2L^3$.
   Top left and bottom right: $L^3R^2LR^2L^3$.
   Top left and top right: $R^2LR^2LR^2L^2R^2L$.

Phase 3: Solve the 2 red squares

1. If the top left square is not red, then turn the right disc to bring a red square to the bottom center of the puzzle, and do $LR^3L^3RLR^3L^3RL$.
2. If the bottom left square is not red, then turn the right disc to bring a red square to the top center of the puzzle, and do $L^3RLR^3L^3RLR^3L^3$.
3. Repeat these steps until both red squares are correct.

Phase 4: Solve the 3 red diamonds

1. If the diamond on the far left of the puzzle is not red, then turn the right disc to bring a red diamond to the far right, and do LR^3, followed by LR^3L^3R ten times, and then RL^3.
2. If the diamond on the top left of the puzzle is not red, then turn the right disc to bring a red diamond to the top right, and do LR^3L^3R ten times.
3. If the diamond on the bottom left of the puzzle is not red, then turn the right disc to bring a red diamond to the bottom right, and do L^3RLR^3 ten times.

2.2.3 Rashkey 3

I will assume that the red pieces belong in the left disk when it is solved. Thus in the mixed puzzle there should be more than two red triangles on the rim of the left disk. If this is not so, you have to turn the puzzle upside down.

Phase 1: Solve all the red triangles This is exactly the same as phases 1 and 2 of Rashkey 2 (but where yellow is mentioned, it should say “non-red”, i.e. “yellow or blue”).

Phase 2: Solve the 6 blue triangles on the rim of the right disc. Note that you can equally well think of this phase as solving the 2 yellow triangles in the center of the left disk.

1. Turn the right disc until there is at least one yellow triangle in the left center.

2. If the yellow triangle is at the top right of the left center, then bring the other yellow triangle in the left center as well by doing the appropriate sequence below:
   Top left: $L^3R^3L^3RL^3R^2L^2RL^2RL^3$.
   Top right: $L^2R^2L^3RL^2R^3LR^2L^2$.
   Right top: $L^3R^2L^3R^2LR^2L$.
   Right bottom: $L^3R^2L^2RLRL^3R^2L^3R^2L^3R^2$.
   Bottom right: $L^3RL^3R^3L^3R^2L^2R^3L^2R^3L^3$.
   Bottom left: $L^2R^2L^3R^3L^2RLR^2L^2$.

3. If the yellow triangle is at the bottom right of the left center, then bring the other yellow triangle in the left center as well by doing the appropriate sequence below:
   Top left: $L^2R^2LRL^2R^3L^3R^2L^2$.
   Top right: $LR^3LRLR^2L^2RL^2RL$.
   Right top: $LR^2L^2R^3L^3R^3LR^2LR^2LR^2$.
   Right bottom: $LR^2LR^2L^3R^2L^3$.
   Bottom right: $L^2R^2LR^3L^2RL^3R^2L^2$.
   Bottom left: $LRLR^3LR^2L^2R^3L^2R^3L$.

Phase 3: Solve the triangles in the right center

- Do one of the following sequences, depending on the position of the two blue triangles in the right center:
  Right: Do nothing as the triangles are solved already.
  Bottom: $RLRL^3R^3LRL^3R^3LR^3$.
  Left: $RL^2R^2L^2R^2L^2R^3$.
  Top: $R^3L^3R^3LRL^3R^3LRL^3R$.
  Top left and bottom right: $R^2L^3R^3L^3RLRLR^3L^3R^3LR$.
  Top right and bottom left: $R^2LRLR^3L^3R^3L^3RLRL^3R$.

Phase 4: Solve the squares

1. If the top left square is not red, then turn the right disc to bring a red square to the top right, do LR^3L^3R five times, and turn the right disc back to its original position.

2. If the bottom left square is not red, then turn the right disc to bring a red square to the bottom right, do L^3RLR^3 five times, and turn the right disc back to its original position.

3. Depending on the positions of the two blue squares, do one of the following move sequences:
Top right, bottom right: Do nothing as they are solved already.
Bottom center, bottom right: Do L^2, then LR^3L^3R five times, and L^2.
Top center, top right: Do L^2, then L^3RLR^3 five times, and L^2.
Bottom center, top center: Do L^3R^3, then LR^3L^3R five times, and RL.

Note that phase 4 uses the sequence $(LR^3L^3R)^5$, which swaps the center squares as well as the top left and top right squares. It possible to solve the squares more quickly than the above method by using different conjugates of that sequence to solve more pieces at the same time. In fact you need to apply that sequence at most twice and often only once is enough.

Phase 5: Solve the diamonds

1. If the left diamond (of the left disc) is not red, then turn the right disc until a red diamond is at the top, and do L, repeat LR^3L^3R ten times, do L^3, and turn the right disc back to its original position.

2. If the top left diamond is not red, then turn the right disc until a red diamond is at the top, repeat LR^3L^3R ten times, and turn the right disc back to its original position.

3. If the bottom left diamond is not red, then turn the right disc until a red diamond is at the bottom, repeat L^3RLR^3 ten times, and turn the right disc back to its original position.

4. Depending on where the yellow diamond is, do one of the following sequences:
Center: Do nothing as the puzzle is solved already.
Top right: Do L^3R^3, then RL^3R^3L ten times, and RL.
Far right: Do L^3R^3, then LR^3L^3R ten times, and RL.
Bottom right: Do LR, then R^3LRL^3 ten times, and R^3L^3.

Note that phase 5 uses the sequence $(LR^3L^3R)^{10}$, which cycles the center, top left and top right diamonds. It possible to solve the diamonds more quickly than the above method by using different conjugates of that sequence to solve more pieces at the same time.

Pretty Patterns for Rashkey 3

- Batman: $R^2LRLRLR^3L^2RL^3R^3L^2RL^2R^3LRLR^3L^3R^3L^3$.
- Butterfly: $L^3RL^2R^3LRLR^2L^3R^2L^2R^2L^2R^2LR^3LR^2LR^3LR^2LR^2$.

3 Kép Korong

Also sometimes called "Rubik's Cheese".

The Kép Korong is a predecessor of the Hockey Puck, and has the same mechanism but with fewer pieces. It has the shape of a thick disk. In the center are two semicircular parts, and around these are 6 segment pieces. The center can rotate with respect to the segments, and one of its halves can be given a 180 degree turn to change the order of the segments. Each side has a picture of a cartoon character. A neat feature is that in its solved state the center must be out of alignment to complete the pictures.

Rubik's Cheese is a predecessor of the Rubik's UFO, and has the same internal mechanism except that it has only one layer instead of two. It is extremely rare and hard to find. It is essentially the same puzzle as the Kp Korong, except that it has no rotating center, so any three adjacent segments can turn at all times. Each segment has different colors on top and bottom, and in the solved position each side of each piece shares its color with a piece next to it. The solved position therefore shows three colors on each side of the puzzle. The picture above shows the version where there are three identical pairs of pieces, but it is more common for the pieces to have six different color pairings so that in the solved position the three colored regions of one layer do not coincide with the three regions of the other.

The US patent for Rubik's Cheese was filed on 9 November 1981 and granted 18 October 1983, US 4,410,179, but there is an earlier Hungarian patent, 9 November 1980, HU 2679.

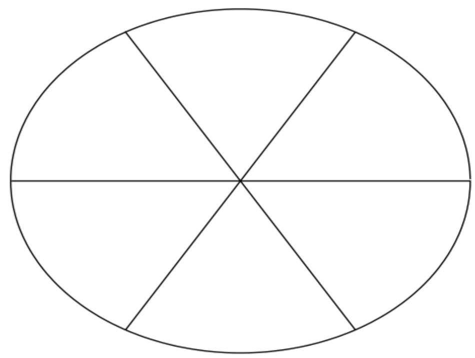

Figure 2: Rubik's chesse

3.1 The number of positions

The pieces come in two sets of three which cannot intermingle. In fact, the Rubik's Cheese has a mechanism where three pieces are fixed on axes connected to a center core, while the other three pieces are held between them. Each piece has two possible orientations, so at first sight there are at most 3!26 = 384 positions. This limit is not reached because the number of flipped pieces of each set of pieces has the same parity as the permutation parity of the other set. This leaves only 3!24 = 96 positions for the Rubik's Cheese.

Moves	Positions
0	1
1	3
2	6
3	12
4	18
5	24
6	23
7	9
Total	96

3.2 Solution

Notation: Mentally label the segments/pieces of the disk clockwise from A to F. Any twist of three segments can then simply be denoted by the letter of the middle segment of those three.

Phase 1: Orient the pieces

1. If there are any three adjacent pieces of which two or three are flipped, then twist that half.
2. Repeat step a as often as possible.
3. If there are still pieces flipped, for example piece A, then do CDE and turn over the puzzle. If necessary, repeat until no flipped pieces are left.

Phase 2: Arrange the segments
1. Consider pieces A, C, and E correct, and compare pieces B, D, and F to them. To cycle around $B \to D \to F$, do the moves CDECDE. To go in the opposite direction $B \to F \to D$ do EDCEDC.

Phase 3: Fix the center (Kp Korong only)

1. During the last move done in the previous phases, you can always ensure that at least one half of the center facing the correct way. Due to parity, the whole center should then be facing the correct way. If you have to turn over the center anyway:
   Do any move (i.e. turn over one half)
   Rotate the center 180 degrees
   Turn over one half again.
2. Rotate the center until the picture on the front is correct.

4 Rubik's Rings / Hungarian Rings

This puzzle consists of two intersecting rings made up of a number of colored balls. The rings of balls intersect at two places, so they share two of the balls. Each ring of balls can be turned, so the balls can be mixed.

The Rubik's Rings version has 34 balls of three colors. The intersections divide each ring into sections; there are 5 balls on the inner sections between the two intersection points, and an outer section with 11 balls. In its solved state the 11 blue, 11 red, 12 yellow balls are arranged so that the outer sections are red or blue, and the inner sections and intersections are yellow.

The Hungarian rings puzzle has 38 balls of four colors, intersections which lie 5 apart (i.e. 4 balls in between them). Two colors have 9 balls (yellow and blue) and two colors have 10 balls (black, red). When solved, the balls of each color must form a continuous row.

Rubik's Rings has the two rings intersecting at an angle, making a 3-dimensional shape. This allows for a neat ratchet system to stop the balls from moving unintentionally.

It might be interesting to quote from the afterword of the Rubik's Cubic Compendium ([**?**], p212) here. It has a picture of the Hungarian rings and the following text by David Singmaster:

> Closer to Rubik's Magic Cube are 'interlocking cycle' puzzles where several rings of pieces cross each other. Endre Pap, a Hungarian engineer, invented a flat version with two rings which was marketed as the Hungarian Rings. The idea was not entirely new, as there is an 1893 patent for it.

That patent is US 507,215 by William Churchill, filed on May 28 1891, granted on October 24, 1893.

4.1 The number of positions of Rubik's Rings

There are 34 balls, which can be arranged in at most 34! ways. This limit is not reached because:

- The yellow balls are indistinguishable (12!)
- The red balls are indistinguishable (11!)
- The blue balls are indistinguishable (11!)
- The red and blue balls are equivalent (2)

The last point is because the puzzle can be solved both with red or with blue on the left hand side. The total number of positions is therefore $34!/(2 \cdot 11!11!12!) = 193,413,243,572,640$.

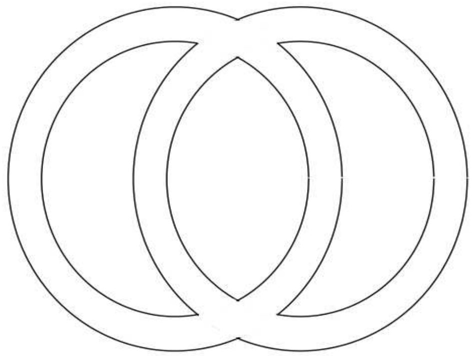

Figure 3: Hungarian rings

4.2 The number of positions of Hungarian Rings

There are 38 balls, which can be arranged in at most 38! ways. This limit is not reached because:

- The yellow balls are indistinguishable (9!)
- The blue balls are indistinguishable (9!)
- The red balls are indistinguishable (10!)
- The black balls are indistinguishable (10!)
- The yellow and blue balls are equivalent (2)
- The red and black balls are equivalent (2)

The total number of positions is therefore $38!/(2{\cdot}9!10!)^2 = 75,406,424,215,922,599,800$. Note however that (even when taking into account the color equivalences) there are still 8 possible solutions.

4.3 Solution to Rubik's Rings

This solution is different to the one in the booklet supplied with the puzzle. I think this solution is quicker. It is also more intuitive and therefore easier to remember.

Phase 1: Solve the yellow balls (the inner area) In this phase all the yellow balls will be placed in position, in the inner sections of the rings including the intersection points.

1. First construct a row of 5 yellow balls in the left ring. This is quite easy to do as follows:

   (a) Find a yellow ball in the ring on the right that does not lie at an intersection.

   (b) Rotate the left ring to bring whatever row of yellow balls you already have into the outer section of the ring, but adjacent to the intersection point nearest your chosen yellow ball.

   (c) Turn the right ring to bring the yellow ball to the intersection, joining it up with the row of balls in the left ring.

   Repeat this process until you have a row of 5.

2. Turn the puzzle upside down so that the row of 5 balls is in the right ring.

3. Now we will put the 7 remaining yellow balls in a row in the left ring. Nearly the same method can be used as in step a, as long as we make sure that the row of 5 remains in the outer section if the right ring and so takes no part in the action. It is however possible that while building your row of 7, all the remaining loose yellow balls also lie in the left ring so that you get stuck at step a1 above. In this case move the left ring so that the yellow row is still in the outer section, but one of the loose yellow balls lies at an intersection point. You can now move the right ring a little and continue with step a2.

4. You now have a row of 5 and a row of 7 yellow balls. First make sure both rows are away from the intersection points, next turn the right ring to put the row of 5 in place between the intersection points, and finally turn the left ring to put the row of 7 in place.

Phase 2: Separate the red/blue balls. In this phase the red and blue colors are separated, thus solving the puzzle. Before you start this phase however, you have to decide which colors the outer sections should be. It is usually best simply to find out which color dominates a section, red or blue, and then consider that to be the color it is to be when solved. Balls of the wrong color in each ring will be called simply 'wrong balls', and balls of the correct color in each ring 'correct balls'. This phase will attempt to swap a wrong ball on the left with a wrong ball on the right, and thus make them correct.

Instead of clockwise or counter-clockwise turns, I will use the terms 'inwards' and 'outwards' turns. An inwards turn of a ring brings some balls from the outer section of the ring inwards towards the top intersection point. In other words, an inwards turn of the left ring will be clockwise but an inwards turn of the right ring will be counter-clockwise. Outwards turns are the opposite of inwards turns.

1. If there is a wrong ball in the top half of each ring, then the following sequence will correct them:

   (a) Turn the right ring inwards to bring a correct ball to the top intersection.

   (b) Turn the left ring inwards to bring the left wrong ball to the top intersection.

   (c) Turn the right ring to bring the right wrong ball to the top intersection.

   (d) Turn the left ring back (outwards) into position.

   (e) Turn the right ring back (outwards) into position.

   Note how at the bottom intersection there will always be a yellow ball, so no changes occur there.

2. Repeat step a as often as you can. Note that as it stands, the wrong ball in the right hand ring can even be in the exact middle of the outer section (6 balls away from the intersection) and the sequence will still work, but the left wrong ball must be above the middle. If the left ball is in the middle and the right hand one is not, then you can use the same sequence but with left and right interchanged in each of the steps a1 to a5.

3. If there are wrong balls in the lower halves of both rings then turn the puzzle upside down and go back to step a to correct them.

4. If there are wrong balls in the top half of one ring, and in the bottom half of another, then this can be solved by bringing one wrong ball to the middle, and going back to step a to correct it. To bring a wrong ball in the top half to the middle, do the following sequence:
   (a) Turn the ring with the wrong ball 6 steps inwards, bringing the ball that was in the middle to the top intersection.
   (b) Turn the other ring inwards till a correct ball lies at the top intersection.
   (c) Turn the first ring outwards till the wrong ball lies at the top intersection.
   (d) Turn the other ring back (outwards) into position.
   (e) Turn the first ring back (outwards) into position.
5. The only case that can not be handled by the previous steps is when there are only two wrong balls, both in the middle. This can be solved by the following sequence:
   (a) Turn one ring inwards 6 steps.
   (b) Turn the other ring inwards 6 steps.
   (c) Turn the first ring outwards 6 steps.
   (d) Turn the other ring outwards 6 steps.

Note: Very often you can run various instances of the sequence in phase 2 step a together. The first turn brings a correct ball to the top intersection, and from then on each turn brings a wrong ball to the intersection. You must take care not to turn too far outwards, as the bottom intersection must always have a yellow ball. When you have no wrong balls within reach, turn the rings back to their original positions. This will speed up the process considerably.

5 Dino Cube / Rainbow Cube / Brain Twist

The Dino Cube is a cube shaped puzzle, and like the Skewb, it has eight axes of rotation centered around the corners. Its cutting planes go diagonally through the square faces, cutting off triangular pyramidal corners. There are twelve moving pieces, one on each edge of the cube. It is called a dino cube because it originally had pictures of dinosaurs on the sides. Versions with 6 colors (pictured above) were also made, as well as with only 4 or 2 colors.

The second puzzle pictured above is the Rainbow Cube. It has the shape of a cuboctahedron. It is very much like a Dino Cube in which the corners have been cut off, giving it 8 triangular faces as well as 6 square ones. The puzzle still has only 12 moving pieces, but now there are also stationary centers in the triangular faces. There are two color schemes. One has 14 colors, the other has only 7 colors with opposite faces the same color. As the puzzle with the 7 color scheme does not have any identical pieces, the two color schemes give puzzles of the same difficulty.

The Brain Twist is a new puzzle by Hoberman. The third picture above shows what it looks like when unfolded in its star shape. To make a move, the points of the star are pushed together causing it to fold into a tetrahedron shape, after which the four corners of the tetrahedron can be twisted. You can unfold it and then fold it in the other direction to get a different tetrahedron with four new corners to twist. The equivalence with the dino cube is most easily seen when the puzzle is in its star shape, and imagining the effect of a twist of three pieces around a corner. There are two solution patterns - one with each tetrahedron face a single color, and one with each tetrahedron corner a single color. The second solution does not look as nice when in star shape however. It was patented by Charles Hoberman and Matthew Davis on 12 May 2005, US 2005/098947.

Jackpot, also known as Platypus, is a new puzzle from Meffert's, and the fourth picture above shows a prototype version. Unlike the other puzzles on this page, on the Jackpot the eight axes are not identical. It is not cube shaped but is based on the tetrahedron, with the corners extended in a triangular cylindrical shape. There are four large hexagonal faces, marked with card suits, as well as four small triangular faces, marked with J, Q, K, and A card values. The pieces are all similarly marked on their sides. For an extra challenge, on some versions of the Jackpot the hexagonal faces also have colored marks along their edges, so that their orientations matter. It was patented by Yusuf Seyhan on 16 January 2003, WO 03/004117.

There is also another puzzle that is equivalent to the Dino Cube, but I do not know much about it. It is in the shape of a Stella Octangula, of which the corners can rotate. There are small pieces in between them along the edges of the internal octahedron, and the three pieces around each tip form a circle on the surfaces of the adjacent tips. Each circle should be of one color.

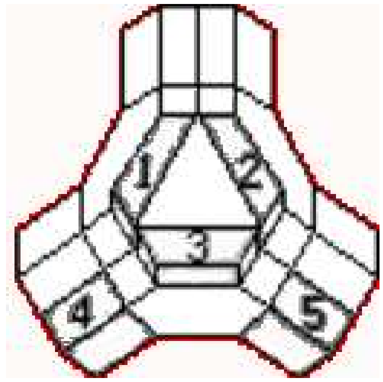

Figure 4: Jackpot

5.1 The number of positions

There are 12 moving pieces, which seemingly have 2 possible orientations giving at most $212 \cdot 12!$ positions. This limit is not reached because

- The pieces cannot actually be flipped (212)
- The pieces must have an even permutation (2)
- On the Dino Cube, Brain Twist, and Stella, the orientation of the puzzle is immaterial (12).

This leaves $11!/2 = 19,958,400$ positions on the Dino Cube, Brain Twist, and Stella, and $12!/2 = 239,500,800$ positions on the Rainbow Cube and standard Jackpot.

Some versions of the Jackpot puzzle have colors on the sides of the hexagonal faces, making their orientation visible. As these faces can be twisted independently, there are 34 times as normal, viz. $34 \cdot 12!/2 = 19,399,564,800$ positions.

The Dino Cube actually has two solutions, which are mirror images of each other. This is similar to the two solutions of the Brain Twist.

The minimal number of moves to achieve each possible position of the Dino Cube/Brain twist needs to solve was computed by the second author (J.S.). There are 1004 positions of the puzzle at maximum distance from start. For the Rainbow cube, the number of moves in each position was calculated by Claude Crpeau and Thanh Vinh Nguyen. There are 366 positions of the puzzle at maximum distance from start. In each case, the tabulations are shown in the table in [Sch2].

5.2 Solution

These are very easy puzzles to solve. There is really only one easy sequence of moves that you need to be able to do. Consider two adjacent axes on the puzzle, with the five piece positions they contain numbered as shown here:

```
1 2
3
4 5
```

You are trying to put the correct pieces at positions 1 and 2, and you haven't yet solved 3, 4 and 5 yet (though it is quite possible that one or more of them happen to be correct, ignore them). Solving the first piece, at position 1 is trivial, since you can just put it there without disturbing previously solved pieces. Next, you want to put the correct pieces at position 2, without disturbing the piece at 1.

1. First bring the piece belonging at 2 to position 4 or 5, without disturbing any of the pieces you have previously solved.
2. Then turn the top face/corner clockwise, putting the piece that currently is at position 2 to the intersection at position 3.
3. Turn the bottom face/corner to bring the piece belonging at 2 to position 3, replacing the old piece.
4. Turn the top face/corner anti-clockwise, back to its previous position, which puts the correct pieces at both 1 and 2.

Once you understand this simple technique, the puzzle can be easily solved by solving the pieces in the following order:

1. Solve three adjacent pieces that share an axis (a triangular face in the Rainbow Cube, three pieces at one corner in the Dino Cube, etc).

2. Solve the six pieces that are adjacent to the first three you solved. You need not move any of the first three solved pieces at all.

3. Solve the last three pieces (which are the ones directly opposite the first three). This is trivial, as you only need at most a single move to do this (though see note below).

Sometimes however it is hard to recognise which piece belongs where. On the Rainbow Cube each color is used on opposite faces of the puzzle, which means that each piece also has a mirror image twin. If you just keep track only of the colors of the triangular faces, then you cannot go wrong as the mirror piece will have its colors swapped if you try to use it instead. On the Dino Cube adjacent pieces have only one color in common, so it seems that you could put the wrong pieces next to each other. This is not the case however - if you try to place the wrong piece, it will seem to be flipped.

On the Brain Twist it is easy to get confused because of the constant need to flip the puzzle inside out, but this puzzle has another neat twist in store. It may happen that you solve the puzzle completely but for two pieces that need to be swapped. This is unfortunately not possible (it is an odd permutation and every move is an even permutation). What has happened is that you have built the color pattern in mirror image. You have to practically re-solve the whole puzzle - choose any face you had solved, swap two of its pieces, and continue from there.

If you have a puzzle on which the face orientations are visible, then the techniques outlined above are not quite sufficient. With a little care you can ensure the face orientations are correct of all but the last face (step c above). If you are solving a Jackpot where the hexagonal faces are marked, then you can simply leave an unmarked triangular face till last instead.

Pretty Patterns and other moves:

Spot Patterns: The rainbow cube has two types of spot patterns, patterns in which only the triangular centers have a different color. One has 8 spots, the other only 6 spots.

- The 8 spot is made like this: Hold the puzzle with a square on top. Make a slice move, i.e. turn any triangular face clockwise, and its opposite face anti-clockwise (so the two faces have actually moved as a unit). Turn the whole puzzle a quarter turn around the vertical axis. Repeat this a number of times. After 6 or 10 times (depending on the direction you turn the puzzle) you will get the 8 spot.

- The 6 spot is done much the same way, except that a triangular face is on top, the slice move is done on any of the other faces, and the puzzle is rotated through 120 degrees each time. After doing this 8 or 10 times, depending on the direction, the 6 spot will appear.

Swapping between solutions The Dino Cube has two solutions which are mirror images of each other. To change a solved Dino Cube to show the other solution, perform the following move sequence:
$urburfurb^{-1}ulfurfdrb^{-1}urb^{-1}ulbdlb^{-1}dlf$. There is a longer method which is much easier to remember:
$(urbulfdrfdlb)(urf^{-1}ulb^{-1}drb^{-1}dlf^{-1})(urbulfdrfdlb)$. This first turns one tetrad of corners clockwise, then the other tetrad counter-clockwise, and then the first tetrad clockwise again. The same method works on the Brain Twist to swap from the faces-solution to the corners-solution, and vice versa. Simply twist all corners clockwise, flip it, twist all corners counter-clockwise, flip again, and turn all corners clockwise.

6 Swissmad

Swissmad is a puzzle with 12 square tiles arranged the shape of a cross. The tiles are contained in a flat frame which has transparent front and back plates, so that both sides of the tiles are visible. The cross has a 2 by 2 square in the center and arms 2 tiles wide and one tile long. The two columns of the vertical arms can slide up or down one tile, independently of each other, by pushing the two parts of the transparent front plate of the puzzle up or down. Similarly, the two rows of the horizontal arms of the cross can individually slide left or right by pushing the transparent back plate sections left or right.

Each tile is red or white on the front and red or white on the back. This gives four possible tile types, and there are exactly three of each type of tile. In the standard solved position the back of the cross is half white and half red, while the front of the cross is in quarters of alternating color.

Embossed on the frame are eight patterns that you can try to make. The names are reproduced below with the names they have been given in the leaflet provided with the puzzle (Time, Interaction, Excellence, Balance, Diversity, Audacity, Mutual help, Security). For example, the solved "Time" puzzle is the pattern

The leaflet quite amusingly tries to justify the names of these patterns. The leaflet itself is also in the shape of a cross, normally with the arms folded

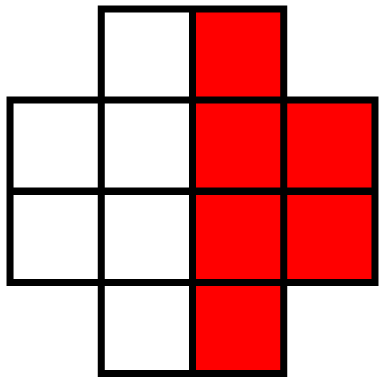

Figure 5: Swissmad's "Time"

over to make it square. The arms of this leaflet show the patterns 'Time', 'Excellence', 'Balance' and 'Diversity', so by changing the order in which you fold these arms, you can have any of these patterns on the front. By some clever folding, you can however also get it to show the 'Mutual help' pattern instead.

Swissmad was invented by Olivier Pahud, and patented on 16 June 2005, WO 2005/053809.

6.1 The number of positions

The packaging states "Two faces, two colors, one million combinations". As you will see, this is not quite accurate. Lets assume the rows and columns are centered. There are 12 pieces, in four sets of identical triplets. These can be arranged in $12!/3!^4 = 369,600$ positions. If you don't assume the rows and columns are centered, then this number must be multiplied by 34 because each row/column has 3 states. Then there are $12!/2^4 = 29,937,600$ positions.

6.2 Solution

There is a very simple sequence of four moves that cycles three pieces around. Slide a column down, a row to the right, the column back up, and finally the row back to the left. The three pieces that are moved are the tile at the intersection of the row and column, as well as the tile above and the tile to the left. Basic move sequence

Any three adjacent tiles that do not lie in a straight line but in a kind of L formation, any such three tiles can be cycled around. Simply shift the row and column alternately to push the two outside tiles towards the middle one, and then move the row and column back. Using this sequence it is quite easy to solve any position.

To solve a position, the best general strategy is to first solve the arms of the cross one by one. With the move sequence you can move any tile from an arm into the central area, then move it around the central area to the side you want, and finally move it out into an arm of the cross. Once all the arms are correct it is usually an easy matter to solve the central area. Always remember to check both sides of the tile are correct for the spot you are moving it to.

One minor complication that can arise with this method is that you may end up needing to swap two pieces without moving anything else. This usually happens when all the arms have been solved, and the central area contains all four types of tiles. In this case replace any tile on an arm by the identical one from the central area. After this, the puzzle can be solved easily.

When solving the standard pattern (time/diversity), I usually do things differently. I solve the 'time' side first, putting all the reds in one half, all the whites in the other. When doing this I don't need to worry about the other side of the puzzle. Then I turn it over and separate each of those halves again, into red and white quarters. By never letting any tile from the other half enter the half you are solving, the 'time' pattern on the back will not be disturbed. The advantage of this method is that you don't need to keep track of the reverse colors of the tiles at all.

7 Diamond 8-Ball Puzzle

This puzzle consists of 8 numbered balls which can roll around a 3×3 square. There is a restriction that differentiates it from a simple 3×3 version of the fifteen puzzle. There are walls on the left and right side of the center position, so that a ball can enter or leave the center in a vertical direction only. In the solved state the center spot is empty, and balls are in clockwise numerical order starting from 1 at the top left corner.

This puzzle was invented by Joshua Frankel, and manufactured by Binary Arts, now called ThinkFun.

1	2	3
4	*	5
6	7	8

Diamond 8-ball puzzle

7.1 The number of positions

If we count only those positions with the gap in the center, then there are 8 pieces which can be any order, giving a maximum of $8! = 40,320$ positions. As with the fifteen puzzle, only even permutations are achievable, so there are actually $8!/2 = 20,160$ positions with the center empty, or $9!/2 = 181,440$ positions with the empty spot anywhere. Note however that there are really two solutions - you could solve it with the 1 ball at the bottom right, and then turn the puzzle upside down.

The second author (J.S.) performed a computer search for this puzzle. The following table shows how many positions there are (with the center empty) for each number of moves from the solved position. A move consists of moving the space around the left or right side in either direction, shifting five pieces in the process. The first column shows the number of positions at each distance from only one of the two solutions, the second if either solution is allowed.

Moves	One solution	Two solutions
0	1	2
1	4	8
2	12	24
3	32	64
4	82	156
5	204	360
6	496	792
7	1,153	1640
8	2,431	3202
9	4,325	5132
10	5,829	5556
11	4,408	2916
12	1,133	306
13	50	2
Total	20,160	20,160

7.2 Solution

Notation: There are 2 loops in the puzzle; the left half and the right half. Denote these by L and R. A clockwise shift of all the five balls in a loop is indicated by the relevant letter (L or R), and an anti-clockwise shift by the letter followed by an apostrophe (i.e. by L^{-1} or R^{-1}).

Phase 1: Solve balls 3,4,5, and 6.

1. Do any moves to bring ball 5 to the top right corner, i.e. the position where 3 will be when solved.
2. If ball 4 lies on the right (directly below the 5, or in the bottom right corner), then do $R^2L^2R^{-2}$ so that 4 lies in the left loop and 5 is again top right.
3. Shift the left loop to bring ball 4 to the top position, and then do R. You now should have ball 4 at the top right, and 5 directly below it.
4. If ball 3 lies at the bottom right corner, then do RLR^{-1}.
5. Shift the left loop to bring ball 3 to the top position, and then do R. You now should have balls 3, 4 and 5 in their correct positions.
6. Shift the left loop to bring ball 6 to the bottom position, where it belongs.

Phase 2: Solve the rest.

1. Find ball 7. Depending on its position, do one of the following sequences:
   top-left: $L^{-1}RL^{-1}R^{-1}LLRL^{-1}L^{-1}R^{-1}$
   top: $RLLR^{-1}LRLLR^{-1}$
   left: $R^{-1}LRLLR^{-1}L^{-1}L^{-1}RL$
2. Find ball 8. Depending on its position, do one of the following sequences:
   top-left: $LR^{-1}L^{-1}RL^{-1}R^{-1}LLRL$
   top: $L^{-1}R^{-1}L^{-1}RL^{-1}R^{-1}LLRL^{-1}$

Nice sequences: $LRLRLRL$: Constructs the second solution, with the 1 at the bottom right corner.

8 Cmetrick

This puzzle consists of a frame containing a 3×3 array of colored balls. The balls are identical, and have 6 colors, arranged like the 6 sides of a cube. If you rotate any ball to the left or right, all three balls in the same row are rotated in the same way. If you rotate a ball up or down, then the balls in the same column rotate with it. The aim is of course to get all the balls in the same orientation, so that they show the same color on the front and sides.

The Cmetrick Mini is just the same as the ordinary Cmetrick, except that it is a 2×2 square, so only has 4 balls. It is of course easier to solve, but also easier to handle because it is easier to keep the balls aligned enough to do the moves.

This puzzle is related to the Rolling Cubes Puzzle in that it has a 3×3 array containing cubes which are reoriented. Mechanically it is somewhat related to the Rubik's Clock, as the nine parts are connected with an internal set of cogs. With regards to difficulty this puzzle is more like the Rubik's Clock than the Rolling Cubes Puzzle.

CMetrick was invented and patented by Dror Rom, 8 January 2004, WO 2004/002587.

8.1 The number of positions

There are 9 cubes, each with 24 possible orientations, so this gives an upper bound of 249 positions. These are not all possible however because of parity restrictions. If you move the second and third columns so that the top row balls all have the same orientation parity, and similar the second and third rows to equalise the left column's parities, then the four balls in the bottom right corner will automatically have the same parity as the rest. This therefore means that the real number of positions is $24^9/2^4 = 165,112,971,264$. Note that there are in fact 24 solutions, so if we consider identical any positions that differ only by recoloring then there are really only $24^8/2^4 = 6,879,707,136$ positions.

The parity restrictions can be worked out by using linear algebra the same way as the Lights Out puzzles, as explained on the Lights Out Maths page. Suppose we ignore the ball orientations, and only keep track of the orientation parities. There are then only six ways that the parities change by a move - changing all the parities in a column or a row. These six ways are not independent, since the sum of all six have no effect, but any five of them are. This means that there are exactly 25 ways to affect the parities, instead of the 29 which you would have if there were no restrictions. This is where the factor 24 comes from in the previous paragraph. There is another way to look at the same thing. Consider four balls in a 22 square. Any move on the Cmetrick will change the parity of zero or two of these balls. The number of balls with odd parity in this square will therefore always remain even. From this it is easy to see that once the balls on the middle row and column of the Cmetrick are known, the parities of the corner balls can be determined. Again, this is the factor 24 above.

The Cmetrick Mini has 4 cubes, each with 24 possible orientations, so this gives an upper bound of 244 positions. On this puzzle there is only one parity restriction, so the number of positions is $24^4/2 = 165,888$. Again there are in fact 24 solutions, so if we consider identical any positions that differ only by recoloring then there are really only $24^3/2 = 6,912$ positions.

Stefan Pochmann has calculated the number of positions at each depth of the Cmetrick, and his results are shown below left. It shows that in the worst case only 15 quarter turns are needed to solve the puzzle. There are $761,436$ such worst-case positions. Below right are the results of the second author (J.S.) for the Cmetrick Mini, showing you need at most 9 quarter turns to solve it. There are 204 such worst-case positions.

8.2 Solution

Notation: A ball can be rolled in four directions, Up, Down, Left, and Right. Quarter turns in these directions will be denoted by the letters U, D, L and R respectively.

Phase 1: Solve the edge balls, forming a cross. The center ball will be considered already solved. In this phase, the balls adjacent to the center will be solved.

1. Consider the ball below the center. In the next few steps this ball will be rolled until it matches the orientation of the center ball.

2. Look at the front color of the center ball, and see where that color is on the below center ball.

3. Depending on where that color is, do one of the following:
   Right side: L

Left side: R
Up side: D R U L
Down side: U R D L
Rear face: R R
This should match up the front colors.

4. Look at the sides of the below center ball, and compare it to the sides of the center ball.

5. Depending on which way the ball needs to be moved to match the center, do one of the following:
   Clockwise: U L D
   Half turn: U L L D
   Anti-clockwise: U R D
   The two balls should now match exactly.

6. Turn the whole puzzle a quarter turn.

7. Repeat steps 1-6 three more times, so that all the balls adjacent to the center match, forming a cross.

Phase 2—: Solve the corner balls.

1. *Consider the ball at the bottom right corner. In the next few steps this ball will be rolled until it matches the orientation of the center ball.*

2. *Look at the front color of the center ball, and see where that color is on the bottom right corner ball.*

3. *Depending on where that color is, do one of the following:*
   Right side: L U R D
   Left side: R U L D
   Up side: D R U L
   Down side: U R D L
   Rear face: R R U L L D
   This should match up the front colors.

4. *If the sides of the bottom right corner ball do not match the sides of the center, then it needs a half turn which can be done by* `L D R R U L`*.* *The ball should have now been solved.*

5. *Turn the whole puzzle a quarter turn.*
6. *Repeat steps a-e three more times, so that all the corner balls match the center, and so the puzzle should be solved.*

You can now optionally reorient all the balls by turning all rows or all columns in the same way.

9 Topspin

The Topspin puzzle is by Binary Arts (now called ThinkFun), but is also sold as No. Crunch under their XEX brand name. It consists of 20 numbered round pieces in one long looped track. You can slide all the pieces of the loop along. There is also a turntable in the loop which can rotate any four adjacent pieces so that they will be in reverse order. This in effect swaps two adjacent pieces and the two pieces on either side of them. The aim is of course to place the pieces in numerical order.

It was invented by Ferdinand Lammertink, and patented on 3 Oct 1989, US 4,871,173.

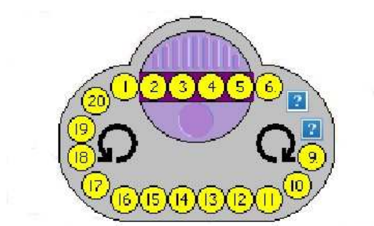

Figure 6: Topspin

9.1 The number of positions

There are 20 distinct pieces, which can therefore be put in at most $20!$ *possible positions. All these are attainable, so there are* $20! = 2,432,902,008,176,640,000$ *distinct positions. If rotations of the loop are considered to be the same, then there are only* $19! = 121,645,100,408,832,000$ *positions.*

9.2 Solution

Phase 1: Place pieces 1-16.

This is very easy. Simply find the next piece to solve and do any move that brings it closer to the solved part of the loop, but which does not move any already solved pieces. Then bring it to a position such that there are three pieces between it and the solved part and then a single turn will append it.

Phase I1: Place pieces 17-20.

1. *Turn the four unsolved pieces if necessary, so that as many pieces as possible are solved. If the same number of pieces is solved either way, then turn it so that piece 17 is as close to piece 16 as possible.*

2. *If exactly two pieces need to be swapped, or their position is ...,16,19,17,20,18,1,2,... which needs 3 swaps to solve it, then the position is an odd permutation. Since a turn is an even permutation this can not be immediately solved. Actually all the pieces that have been solved so far are in the wrong position, and all need to be moved an odd number of places along the loop. The easiest way to do this by repeatedly turning and shifting the loop one step anti-clockwise, until all the solved pieces have been moved (i.e. 16 times). Note that the built-in solver of the JavaScript simulation above will never encounter this problem, since it checks the permutation parity at the start and moves piece 1 one step if the parity is odd.*

3. *To cycle 3 pieces around, simply follow the moves shown in the table below.*

14	*15*	*16*	*19*	*17*	*18*	*20*
14	*17*	*19*	*16*	*15*	*18*	*20*
16	*19*	*17*	*14*	*15*	*18*	*20*
16	*15*	*14*	*17*	*19*	*18*	*20*
17	*14*	*15*	*16*	*19*	*18*	*20*
17	*14*	*18*	*19*	*16*	*15*	*20*
17	*16*	*19*	*18*	*14*	*15*	*20*
17	*16*	*15*	*14*	*18*	*19*	*20*
14	*15*	*16*	*17*	*18*	*19*	*20*
14	*15*	*16*	*17*	*20*	*18*	*19*
14	*15*	*18*	*20*	*17*	*16*	*19*
14	*17*	*20*	*18*	*15*	*16*	*19*
14	*17*	*16*	*15*	*18*	*20*	*19*
14	*18*	*15*	*16*	*17*	*20*	*19*
14	*18*	*15*	*19*	*20*	*17*	*16*
14	*18*	*17*	*20*	*19*	*15*	*16*
14	*18*	*17*	*16*	*15*	*19*	*20*
14	*15*	*16*	*17*	*18*	*19*	*20*
14	*15*	*16*	*18*	*19*	*17*	*20*
18	*16*	*15*	*14*	*19*	*17*	*20*
18	*16*	*17*	*19*	*14*	*15*	*20*
18	*14*	*19*	*17*	*16*	*15*	*20*
18	*14*	*15*	*16*	*17*	*19*	*20*
16	*15*	*14*	*18*	*17*	*19*	*20*
16	*17*	*18*	*14*	*15*	*19*	*20*
14	*18*	*17*	*16*	*15*	*19*	*20*
14	*15*	*16*	*17*	*18*	*19*	*20*
14	*15*	*16*	*17*	*19*	*20*	*18*
14	*19*	*17*	*16*	*15*	*20*	*18*
14	*19*	*17*	*18*	*20*	*15*	*16*
14	*19*	*15*	*20*	*18*	*17*	*16*
14	*19*	*15*	*16*	*17*	*18*	*20*
14	*17*	*16*	*15*	*19*	*18*	*20*
14	*17*	*18*	*19*	*15*	*16*	*20*
14	*15*	*19*	*18*	*17*	*16*	*20*
14	*15*	*16*	*17*	*18*	*19*	*20*

To make two swaps of adjacent pieces, then follow this sequence:

15	16	18	17	20	19
17	18	16	15	20	19
17	20	15	16	18	19
16	15	20	17	18	19
16	15	19	18	17	20
18	19	15	16	17	20
18	17	16	15	19	20
15	16	17	18	19	20

Phase 3: Setting the turntable
If you wish you can turn around the turntable without disturbing the pieces by using the move sequence below.

1	2	3	4	5	6	7
1	5	4	3	2	6	7
3	4	5	1	2	6	7
3	4	6	2	1	5	7
2	6	4	3	1	5	7
2	1	3	4	6	5	7
4	3	1	2	6	5	7
4	3	5	6	2	1	7
6	5	3	4	2	1	7
6	5	3	7	1	2	4
7	3	5	6	1	2	4
7	3	2	1	6	5	4
1	2	3	7	6	5	4
1	2	3	4	5	6	7

10 Zauberkreuz / Magic Cross

This puzzle has a red frame in the shape of a cross containing 20 tiles. The center of the cross 2×2 *tiles large. The vertical arms contain a plunger with a block of* 6×2 *tiles which can slide up or down as a whole by two steps from its central position, so that all its tiles can reach the center* 2×2 *square. The* *horizontal arms similarly have a* 2×6 *block of tiles that can be slid sideways two tiles in either direction.*

Every tile is split in half diagonally, one half colored, the other half blank. This can be done in four different orientations, and indeed there are four tiles of each color in the puzzle. Five colors are used, namely red, yellow, blue, green and purple, making up the 20 tiles of the puzzle. In the solved position the four tiles of each color together form a diagonal square.

This puzzle was sold on the 125th anniversary of the Red Cross (i.e. 1988) and they received part of the proceeds. It was invented by Athanasios Margaritis, patented on 20 October 1988, DE 3,711,368.

10.1 The number of positions

There are twenty pieces which can be arranged in at most 20! positions for any given position of the plungers. In fact, only the even permutations are possible. There are 25 plunger positions, so there are $25 \cdot 20!/2 = 30,411,275,102,208,000,000$ *positions all together.*

10.2 Solution

Notation*: Sliding the horizontal plunger left or right by exactly one tile is denoted by L or R respectively. In the same way moving the vertical plunger up or down is denoted by U or D.*

Phase 1: Solve one square. It is fairly easy to move a tile around without destroying previously placed pieces, so we will skip some of those details in the first two phases. It is actually possible to skip the first two phases all together, and use the techniques in phases 3 and 4 instead, but that is a lot slower.

1. *. Put the plungers in the central position.*
2. *Choose one color, lets say red, and find the tile with the red triangle on the bottom left. Note that this tile belongs at the top right of a square when it is solved.*
3. *Move that tile to the bottom right position in the right hand arm of the cross.*

4. *Find the tile with the red triangle on the top left. Note that this tile belongs at the bottom right of a square when it is solved.*
5. *If that tile happens to lie at the top right of the right arm then do* `LL DD RULUR DD LU RR U` *to swap the two tiles, putting them into their correct positions.*
6. *Otherwise, bring that tile to the bottom left of the right hand arm, without disturbing the first tile. The two tiles can then be solved with the moves* `L DLUR R`*.*
7. *The next steps are very similar to steps b-f. Find the tile with the red triangle on the bottom right. Note that this tile belongs at the top left of a square when it is solved.*
8. *Move that tile to the bottom left of the right hand arm of the cross.*
9. *Find the tile with the red triangle on the top right. Note that this tile belongs at the bottom left of a square when it is solved.*
10. *If that tile happens to lie at the top left of the right hand arm of the cross then do* `L DD RULUR DD LURU` *to swap the two tiles, putting them into their correct positions.*
11. *Otherwise, bring that tile to the bottom right of the center square, without disturbing the first three tiles. The two remaining tiles can then be solved with the moves DLUR.*

Phase 2: Solve another square.

1. *Turn the whole cross so that the arm with the previously solved square lies at the bottom.*
2. *Use the method of phase 1 to solve another square, without disturbing the previously solved tiles.*

Phase 3: Separate the colors.

1. *Choose one of the unsolved colors. You will make the center square this color, so it is best to choose the color that already occurs most often there.*

2. *Find any tile of the chosen color that is not yet in the center square. Hold the cross so that this tile lies in the left arm of the cross. Now note the position of some center tile that is of the wrong color.*

3. *To swap the two tiles, choose the move sequence from the following table that matches the positions of the two tiles.*

Left Square	*Center Square*	*Move Sequence*
Top Left	*Top Left*	$URDLD^2RU^2RULD^3LU^2$
Top Left	*Top Right*	$URDRULURDLDLURULD^2$
Top Left	*Bottom Left*	$URDLURDRUL^2DRULD$
Top Left	*Bottom Right*	$URDLURULDRDRULURDL\ LD$
Top Right	*Top Left*	$DRULDRURULD^2LU$
Top Right	*Top Right*	$URULDRDRULURDLDL$
Top Right	*Bottom Left*	$DRU^3RDLD^2LU^2RULD^2$
Top Right	*Bottom Right*	$DRULDR^2ULDLURDLU$
Bottom Left	*Top Left*	$DRULDRURDL^2URDLU$
Bottom Left	*Top Right*	$DRULDRDLURURDLDRUL^2U$
Bottom Left	*Bottom Left*	$DRULU^2RD^2RDLU^3LD^2$
Bottom Left	*Bottom Right*	$DRURDLDRULULDRDLU^2$
Bottom Right	*Top Left*	$URD^3RULU^2LD^2RDLU^2$
Bottom Right	*Top Right*	$URDLUR^2DLULDRULD$
Bottom Right	*Bottom Left*	$URD^2RULULDRDLU$
Bottom Right	*Bottom Right*	$DRDLURURDLDRULUL$

4. *Repeat steps 2-3 until the center is of one color.*

5. *If two of the arms of the cross still have a mix of colors, then do* $R^2D^2L^2U^2$ *to bring one of them (the left arm) to the center, and go back to step 1. Eventually each arm of the cross will have only one color.*

Phase 4: Solve the squares.

1. *If the center already has a solved square, then hold the cross so that an unsolved square is in the left arm of the cross and do* $R^2D^2L^2U^2$*, so that the center now has one of the unsolved squares.*

2. *Examine the pattern formed by the center tiles, and look it up in the table below. If the pattern is listed, then the move sequence in the table can be used to solve it. You may need to rotate the cross to get the pattern to match.*

Move Sequence	*Effect*
$LU^2RDLD^3RULU^2RULD^3RDLU^2R$	*Diagonal swaps*
$LDLURDLURDLUR^2DLURDLURDLUR$	*Swap top with bottom*
$RURDL^2DLDRU^2R^2ULDLD^2LURU$	*Cycle clockwise all but bottom right*
$DLDRU^2RURDL^2D^2LURUR^2ULDL$	*Cycle anti-clockwise all but bottom right*

Note that these patterns are all even permutations.

3. *If the center pattern could not be found in the table (i.e. it is an odd permutation), then one of the arms of the cross will also have a mixed pattern that is not in the table. Hold that arm of the cross on the left, do* $LDRURURDL^2DLUR^2ULD$*, and go back to step 2.*

4. *Repeat steps 1-3 until there is a square of every color.*

Phase 5: Rearrange the squares.

1. *. Decide which color square you want to have in the bottom arm of the cross. Depending on where that color currently resides, do one of the following sequences:*
   Top: $D^2R^2U^4L^2D^2$
   Left: $U^2R^2D^2L^2$
   Right: $U^2L^2D^2R^2$
   Center: $L^2U^2R^2D^2$
   Bottom: -

2. *Decide which color square you want to have in the right arm of the cross. Depending on where that color currently resides, do one of the following sequences:*
   Top: $L^2D^2R^2U^2$
   Left: $R^2D^2L^4U^2R^2$
   Center: $D^2L^2U^2R^2$
   Right: -

3. *Decide which color square you want to have in the top arm of the cross. Depending on where that color currently resides, do one of the following sequences:*
   Left: $D^2R^2U^2L^2$.
   Center: $R^2D^2L^2U^2$.
   Top: -

4. *Finally, if you wish to swap the center and the left arm of the cross, do* $RU^2LD^2RU^2LD^2RRU^2LD^2RU^2LD^2L$.

11 Turnstile, Puzzler

The Turnstile puzzle consists of two overlapping circular disks. Each disk has 12 pieces, 6 are triangular ('corners'), 6 rectangular ('edges') alternating along the rim of the disk around a hexagonal center. The pieces have curved sides. The disks have three pieces in common, two corners and one edge, so there are altogether 10 corners and 11 edges. Each disk can be rotated. A rotation should be a multiple of 60 degrees since only then can the other disk move again.

There are 5 bright colors used for the corners, a pair of each color. There is one edge of each of those colors, and the remaining 6 edges are grey. The aim is to put together all pieces of the each color. The only way to do this is by forming colored 'lozenges' of two adjacent corners and the edge between them. These lozenges are separated from each other by the grey edges. The rim of the puzzle has small colored markings, so that the colors of the lozenges should be such that they match the markings.

Turnstile was manufactured by Binary Arts (now called ThinkFun). There is also a version called TwinSpin that was made in the far East. Turnstile was licensed however from Douglas A. Engel who originally invented and produced it. He patented it on 15 November 1983, US4,415,158. He first called it Engel's Enigma, but later marketed it as The Puzzler with three different designs of varying difficulty:

Puzzler variations

1. *Novice: The pieces of one disc are blue, the rest red.*

2. *Challenger: The top half of the discs is green, the bottom half yellow, and the three edge pieces in the middle are black.*

3. *Avenger: Like Turnstile, but with the colors yellow, red, green, blue and black, and using white edges to separate them. There are no color markings on the puzzle rim.*

Note that the Puzzlers sold in the USA were manufactured there and have a hexagonal white base, whereas those sold in the rest of the world were made in Taiwan and have a black oval base.

These puzzles were invented by Douglas A. Engel,

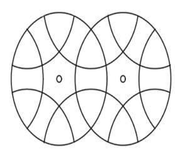

Figure 7: Turnstile

11.1 The number of positions

There are 10 corners and 11 edges, so there are at most $11!10!$ *positions. This limit is not reached on any of these variations because there are many sets of identical pieces. The following table shows the numbers for each variation.*

Puzzle	*Corners*	*Edges*	*Positions*
Puzzler Novice	*4, 6*	*5, 6*	$10!11!/(4!5!6!^2) = 97,020$
Puzzler Challenger	*5, 5*	*3, 4, 4*	$10!11!/(3!4!^2 5!^2) = 2,910,600$
Puzzler Avenger	*2, 2, 2, 2, 2*	*1, 1, 1, 1, 1, 6*	$10!11!/(2!^5 6!) = 6,286,896,000$

Remember however that the Puzzler avenger actually has $5! = 120$ *solutions due to the lack of markings. Equivalently you can consider the positions that differ by a permutation of colors to be the same, and then it has one solution but only* $10!11!/(2!^5 6!5!) = 52,390,800$ *positions.*

The second author has done a computer analysis of the Puzzler Novice and the Puzzler Challenger in order to find God's Algorithm for them. The results are in the tables below. Analogous to the Rubik's cube, there are two ways to count the moves. The Face Turn Metric means that a turn of either disk by any amount is a single move. The "Quarter" Turn Metric means that only 60 degree turns are single moves. The tables shown in [Sch2] that they both can be solved in no more than 13 face turns, or that they need at most 18 or 19 sixth turns respectively.

11.2 Solution to Turnstile and Puzzler Avenger

Notation*: Let a clockwise 60 degree rotation of the left disk be denoted by L. Rotations of 120, 180, 240, 300 degrees are then denoted by* L^2, L^3, L^4 *and* L^5. *Note that* L^5 *can also be considered an anti-clockwise 60 degree turn, and is therefore also denoted by* L^{-1}. *Turns of the right disk are denoted in the same way, but using the letter R.*

A solution for any variation of the puzzle is fairly easily constructed by the use of the simple move sequence $LR^{-1}L^{-1}R$. *This sequence swaps two pairs of corners, and also does a 3-cycle of edges. If you perform it twice then the corners remain unmoved but it is still a 3-cycle of edges. If you perform it three times then the edges remain unmoved but still does a double swap of corners, though if you solve the corners before solving the edges you don't need to do this. Below, we will explain a solution to the standard Turnstile and Avenger puzzles which uses some shortcuts instead of applying this move sequence all the time.*

Phase 1: Pair up the corners.

1. *First place two corners of the same color next to each other. This is very easy. If there is not already a pair, turning only one of the disks is nearly always sufficient.*

2. *Rotate the pair to the overlapping region, and then do L2 to place them at the bottom left of the left disk.*

3. *Find the corner matching the color of the upper left corner. If it does not lie at the top of the left disk (i.e. it is not yet paired up) then it must lie in the right disk. Turn the right disk to bring it to the top and do* $LR^{-1}L^{-1}$. *The left disk must now have two corner pairs.*

4. *If the right disk has one or more pairs then rotate it to bring a pair to the overlapping region, otherwise rotate it so that its top and bottom corners have the same color.*

5. *Only 4 possibilities remain for the corners other than the three pairs position.*

   (a) *If there are two pairs on the right disk then do* $R^{-1}LRLR$.

   (b) *If there is one pair on the right disk then do* $R^3L^4R^3L^2R$.

   (c) *If each pair of corners lie on the opposite sides of the disk then do* $L^2R^2L^2R^2L$.

   (d) *. Otherwise do* $R^2L^2R^3L^{-1}RL$.

Phase 2: Place the edges correct.

1. *Find a colored edge which is in a spot where a grey edge belongs (i.e. it does not lie between a matched pair of corners).*

2. *Find the corner pair of the same color as the edge.*

3. *The following steps should bring the corners to the overlapping region and the edge adjacent to it.*

   (a) *If the edge and its corners lie in the same disk then rotate it to bring the corners to the overlapping region. If the edge is not yet adjacent to the corners then rotate the other disk 120 degrees and go to step 2.*

   (b) *If the edge and its corners lie in different disks and are not in the overlapping region then rotate the edge to a position adjacent to the overlap and then turn the other disk to place the corners in the overlap.*

   (c) *The edge can now be placed between the corners by using one of the following four similar sequences TWICE. Which one you need depends on the position of the edge piece:*
   Edge is at top left: $R^{-1}LRL^{-1}$.
   Edge is at top right: $LR^{-1}L^{-1}R$.
   Edge is at bottom left: $RL^{-1}R^{-1}L$.

Edge is at bottom right: $L^{-1}RLR^{-1}$.
Repeat step 1-4 for all the edges. If all the grey edges are correct but the colored ones are not, then you can use the procedure above to insert a grey edge in an incorrectly colored 'lozenge' to free up a colored edge for step 1.

Phase 3: Place the lozenges correct.

1. *This phase mostly uses only 120 degree turns of the discs, so that the lozenges will not broken be up. First put the correct lozenge in bottom left position which is trivial.*
2. *Find the lozenge belonging at the top left position. If it is not yet in position then rotate it to the top right, and do* $L^2R^4L^4$.
3. *Rotate the right disc to bring the correct lozenge to the overlapping region.*
4. *If the final two colors are not correct then we must break them up and rebuild them by performing the sequence* $LRL^{-1}RLR^4L^{-1}$ *to swap the corners and then solve the remaining edge as in phase 2 by using:* $R^2LR^{-1}L^{-1}RLR^{-1}L^{-1}R$.

Pretty Patterns for Puzzler:
1. $LRLRLRLR^3LR^{-1}L^{-1}R^{-1}$
2. $LR^{-1}LR^{-1}LR^{-1}LR^{-1}LR^{-1}$
3. $LRL^2R^{-1}L^{-1}RL^{-1}R^{-1}LR^{-1}LR^4$
4. $L^4R^{-1}LRL^3RL^{-1}RL^{-1}RL^{-1}RL^2RL^{-1}$
5. $RLR^4LRL^{-1}R^2L$

References

[Sch2] Jaap Scherphuis' puzzle page, on the WWW at the URL: `http://www.geocities.com/jaapsch/puzzles/`

[Ru] E. Rubik, et al, **Rubik's cubic compendium**, *Oxford Univ Press, 1987.*